Recall $\mathcal{M}_n / \mathbb{Z}[\frac{1}{n}]$ functor of $(E,\alpha)/\cong$ level-n-str.

Lemma Assume $\mathcal{M}_n[\frac{1}{a}] / \mathbb{Z}[\frac{1}{an}]$ & $\mathcal{M}_n[\frac{1}{b}] / \mathbb{Z}[\frac{1}{bn}]$ representable, $(a,b)=1$.

Then $\mathcal{M}_n$ representable.

If $\mathcal{M}_n[\frac{1}{a}]$, $\mathcal{M}_n[\frac{1}{b}]$ affine, then so is $\mathcal{M}_n$.

Proof Gluing. $\square$

·) Our aim is to construct $\mathcal{M}_n[\frac{1}{3}]$ using known representability of $\mathcal{M}_3$.

(Case of $\mathcal{M}_n[\frac{1}{2}]$ same using $\mathcal{M}_4$ instead.)

·) Have projection maps $\mathcal{M}_{mn} \longrightarrow \mathcal{M}_n$

$$(E,\alpha) \longmapsto (E, m\cdot\alpha)$$

Relies on: If $\alpha_1, \alpha_2 \in (\mathbb{Z}/nm)^{\oplus 2}$ basis as $\mathbb{Z}/nm$-module, then $m\alpha_1, m\alpha_2 \in (\mathbb{Z}/n)^{\oplus 2}$ basis as $\mathbb{Z}/n$-module.

Lem $\mathcal{M}_n$ representable $\Longrightarrow$ $\mathcal{M}_{m\cdot n}$ representable $\forall m \geq 1$.

Proof Given $(E,\beta)/S$ EC + level-n-str,

$\exists$ fin étale $L_{(E,\beta),nm} \longrightarrow S$ $\quad (nm \in \mathcal{O}_S^\times)$

$$\mathcal{L}_{(E,\beta),nm}(T) = \{ \text{level-}n\cdot m\text{-str } \alpha \text{ for } E_T \text{ s.th. } m\cdot\alpha = \beta_T \}$$

Indeed, $\mathcal{L}_{(E,\beta),nm} \subseteq \mathcal{L} := \mathcal{L}_{E,nm}$ open + closed subscheme where $m\cdot\alpha = \beta_T$ holds:

$$\begin{tikzcd} E_{\mathcal{L}}[nm] \times_{\mathcal{L}} E_{\mathcal{L}}[nm] \arrow[d, "\text{fin ét}"'] \\ \mathcal{L} \arrow[u, bend right, "m\alpha_1,\, m\alpha_2,\, \beta_1,\, \beta_2"'] \arrow[d, "\text{fin ét}"'] \\ S \end{tikzcd}$$

$\alpha_1, \alpha_2 :=$ universal level-nm-str

$\underbrace{m\alpha_1, m\alpha_2, \beta_1, \beta_2}_{\text{fin. ét. by Principle 1}}$

$$\mathcal{L}_{(E,\beta),nm} = \mathcal{L}_{(m\alpha_1, m\alpha_2),\, E_{\mathcal{L}}[nm] \times_{\mathcal{L}} E_{\mathcal{L}}[nm],\, (\beta_1,\beta_2)} \cap \mathcal{L}$$

$\cap\!|$ $\mathcal{L}$ ⟵ open + closed subscheme via any projection.

Apply this to universal pair $(E,\beta)/\mathcal{M}_n$. $\square$

Just as $\mathcal{L}_{E,n} \to S$ $(n \in \mathcal{O}_S^\times)$ before,

$\mathcal{L}_{(E,\beta),nm} \to S$ is torsor for
$$K_{nm\to n} := \ker\left(GL_2(\mathbb{Z}/nm) \to GL_2(\mathbb{Z}/n)\right)$$

Cor $\forall n$, $M_{3n}/\mathbb{Z}[\frac{1}{3n}]$ is representable by affine scheme.

Situation

$$\begin{tikzcd}
 & M_{3n} \arrow[dl, "q"'] \arrow[dr, "f"] & \\
Y_n := K_{3n\to n} \backslash M_{3n} \arrow[rr, shift left, "\Phi"] & & M_n \arrow[ll, shift left, "\Psi"]
\end{tikzcd}$$

•) Assume $n \geq 3$. Then $K_{3n\to n} \circlearrowright M_{3n}$ freely. (Highly recommended) exercise.

$\Rightarrow$ q is a $K_{3n\to n}$-torsor, in partic finite étale.

•) Let $y \in Y_n(S)$. $\exists\, S' \xrightarrow{u} S$ fin et

+ $(E,\alpha) \in M_{3n}(S')$ s.t. $q(E,\alpha) = y \circ u$.

E.g. take $S' := M_{3n} \times_{q, Y_n, y} S$

Then $f(E,\alpha) = (E, 3\cdot\alpha =: \beta) \in M_n(S')$

Would like $(E,\beta) \in M_n(S)$ in unique way,

then put $\Phi(y) = (E,\beta)$.

·) Conversely $(E,\beta)/S$, $\exists S' \to S$ fin ét

+ level $3n$ str. α for $E_{S'}$ s.t. $3\cdot\alpha = \beta_{S'}$.

Would like $q(E_{S'},\alpha) \in Y_n(S)$ in unique way,

then put $\Psi(E,\beta) = q(E_{S'},\alpha)$.

§ Descent References: Vistoli "Grothendieck topologies, fibered categories, ..."

in FGA explained

BLR §6.

$R \xrightarrow{p^*} R'$ ring map, M R-module.

Have two maps $p_1^*, p_2^* : R' \to R'' := R' \otimes_R R'$

+ natural iso

$$R'' \otimes_{p_1^*, R'} (R' \otimes_R M) \xrightarrow{\cong} R'' \otimes_{p_2^*, R'} (R' \otimes_R M)$$

$$(1\otimes 1)\otimes(1\otimes m) \longmapsto (1\otimes 1)\otimes(1\otimes m)$$

Get functor $R\text{-Mod} \xrightarrow{p^*} \left\{ (M', \phi) \;\middle|\; \begin{array}{l} M' \ R'\text{-mod} \\ \phi : p_1^* M' \xrightarrow{\cong} p_2^* M' \end{array} \right\}$

Thm (Grothendieck, BLR Thm 6.1.4)

Assume $R \to R'$ faithfully flat. Then

1) p^* is fully faithful
2) Essential image is those (M', ϕ) that satisfy cocycle condition

$$\begin{tikzcd}
p_1^* M' \arrow[rr, "p_{12}^* \phi"] \arrow[dr, "p_{13}^* M'"'] & & p_2^* M' \arrow[dl, "p_{23}^* M'"] \\
& p_3^* M' &
\end{tikzcd}$$

p_i, p_{ij} the projections $\operatorname{Spec} R''' \to \operatorname{Spec} R'', \operatorname{Spec} R'$

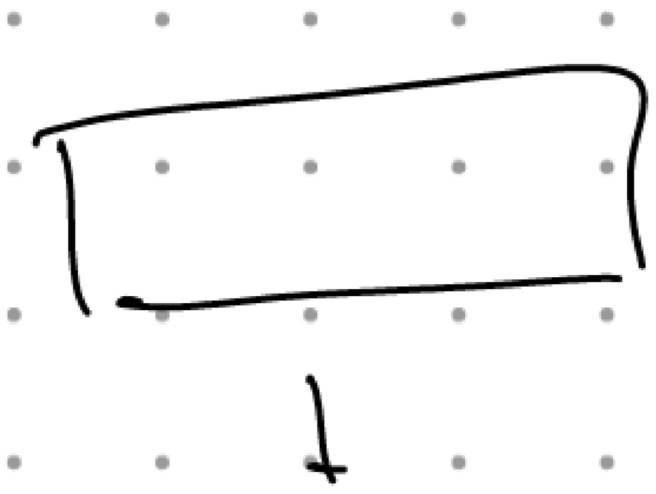
Quasi-inverse for p^*:

$$M = \{ m' \in M' \mid \phi(1 \otimes m') = 1 \otimes m' \}$$

Most important input in proof:

$$R = \{ r' \in R' \mid r' \otimes 1 = 1 \otimes r' \text{ in } R'' \}.$$

Intuition



$$\begin{array}{c} X \\ \downarrow \\ S \end{array}$$

functions on S
= fiber-wise const. fcts. on X
= functions f on X
s.t. $f \circ p_1 = f \circ p_2$.
for $X \times_S X \rightrightarrows X$.

[BLR §6.2]

<u>Example</u> $(f_1, \dots, f_n) = 1$ in R $\iff$ $\operatorname{Spec} R = \bigcup_i D(f_i)$

1) $R \longrightarrow R' := \prod_{i=1}^{n} R[f_i^{-1}]$ faith. flat.

$R'' = \prod_{i,j} R[(f_i f_j)^{-1}]$, $R''' = \prod_{i,j,k} R[(f_i f_j f_k)^{-1}]$

and Thm becomes Zariski gluing of q-coh. modules.

2) $\Gamma \curvearrowright R'/R$ finite étale <u>Galois extension</u>.

$\underset{\text{def}}{=}$ Γ finite group $\curvearrowright R'$ s.th. $\prod_{\gamma \in \Gamma} R' \xrightarrow{\cong} R' \otimes R'$

$\gamma : R' \xrightarrow{\cong} R'$ $\qquad (r')_\gamma \longmapsto (r', \gamma(r'))$

Then $R'' = \prod_{\gamma \in \Gamma} R'$, $\quad R''' = \prod_{\gamma_1, \gamma_2} R'$.

Datum $\phi : R'' \otimes_{R'} M' \longrightarrow R'' \otimes_{R'} M'$

same as a family $\left\{ \phi_\gamma : R' \otimes_{\gamma, R'} M' \xrightarrow{\cong} M' \right\}$

$$\begin{tikzcd} & R' \otimes_{\gamma,R'} M' \arrow[r, "\phi_\gamma"] & M' \\ 1 \otimes m' & & \\ m' \arrow[u, mapsto] & M' \arrow[uu, "\cong"'] \arrow[uur, "\psi_\gamma"'] & \end{tikzcd}$$

Then ϕ_γ R'-linear $\iff$ ψ_γ γ-semilinear

$$\psi_\gamma(r' \cdot m) = \phi_\gamma(1 \otimes r' \cdot m) = \phi_\gamma(\gamma(r') \otimes m)$$
$$= \gamma(r') \cdot m$$

[BLR §6.2.B]

May check: $\{\phi_\gamma\}_{\gamma \in \Gamma}$ satisfy cocycle condition

$\iff$ $\{\psi_\gamma\}_{\gamma \in \Gamma}$ give group action $\Gamma \circlearrowright M'$

Thm specializes to $\quad R\text{-Mod} \cong \left\{ (M', \psi) \;\; \begin{matrix} M' \ R'\text{-module} \\ + \psi : \Gamma \circlearrowright M' \end{matrix} \right\}$

Γ-action s.th. $\gamma(r'm) = \gamma(r') \cdot \gamma(m)$.

Functor $\quad M \longmapsto R' \otimes_R M \; + \; \gamma(r' \otimes m) = \gamma(r') \otimes m$.

<u>Example</u> $\mathbb{C}/\mathbb{R}$, σ Galois conj. $\mathbb{G}_{m,\mathbb{C}} = \operatorname{Spec} \mathbb{C}[t^{\pm 1}]$.

$$\begin{tikzcd}
\mathbb{G}_{m,\mathbb{C}} \arrow[r, "\sigma"] \arrow[d] & \mathbb{G}_{m,\mathbb{C}} \arrow[d] \\
\operatorname{Spec} \mathbb{C} \arrow[r, "\sigma"] & \operatorname{Spec} \mathbb{C}
\end{tikzcd}
\quad \text{as} \quad \sigma(\lambda t^n) = \sigma(\lambda) \cdot t^{-n}$$

Is descent datum by Example 2. Compatible w/ group str on $\mathbb{G}_{m,\mathbb{C}}$ $\Longrightarrow$ $(\mathbb{G}_{m,\mathbb{C}}, \sigma)$ descends to affine group scheme $/\mathbb{R}$.

It is $\quad \operatorname{Spec}\left(\mathbb{C}[t^{\pm 1}]\right)^{\sigma = id}$.

$$= \operatorname{Spec}\left(\mathbb{R}[t + t^{-1}, i(t - t^{-1})]\right)$$

$$\cong \operatorname{Spec}\left(\mathbb{R}[X,Y] / (X^2 + Y^2 - 1)\right)$$

$X = \frac{t+t^{-1}}{2}$, $Y = \frac{i(t-t^{-1})}{2}$

Represents functor:

$$R/\mathbb{R} \longmapsto \{x \in \mathbb{C} \otimes_{\mathbb{R}} R \mid x \cdot (\sigma \otimes 1)(x) = 1\}$$

Torus of elements of norm 1 in $\mathbb{C}$.

§ fpqc morphisms

Def $S' \xrightarrow{p} S$ <u>fpqc</u> $\overset{=}{\text{dif}}$ flat, surjective

+ $\forall\ U \subseteq S$ open quasi-compact

$\exists\ V \subset S'$ open quasi-compact w/ $p(V) = U$.

Example p flat, surj, quasi-compact.

$S'' := S' \times_S S'$, $S''' := S' \times_S S' \times_S S'$.

Formalities like cocycle condition etc. carry over.

Thm $p : S' \to S$ fpqc

$$\mathrm{Qcoh}(S) \xrightarrow{\cong} \left\{ (\mathcal{F}', \phi) \;\middle|\; \begin{array}{l} \mathcal{F}' \text{ qcoh } \mathcal{O}_{S'}\text{-mod} \\ \phi : p_1^* \mathcal{F}' \xrightarrow{\cong} p_2^* \mathcal{F}' \\ \text{s.th. cocycle cond holds} \end{array} \right\}$$

Idea $U = \operatorname{Spec} R \subseteq S$ affine.

Choose $V \subseteq S'$ qc s.th. $p(V) = U$.

Write $V = \bigcup \operatorname{Spec} R_i'$, put $R' = \prod R_i'$

Then $V \times_U V = \bigcup \operatorname{Spec} R_i' \otimes_R R_j'$, $R'' = \prod R_i' \otimes_R R_j'$.

Zariski descent for qcoh sheaves allows to reduce to affine diagrams $R \to R' \rightrightarrows R''$.

<u>Cor</u> $p: S' \to S$ fpqc. Then

$$\{X \to S \text{ affine}\} \xrightarrow[p^*]{\cong} \left\{ (X' \to S', \phi) \;\; \begin{array}{l} X'/S' \text{ affine} \\ \phi: p_1^* X' \xrightarrow{\cong} p_2^* X' \\ + \text{cocycle} \end{array} \right\}$$

<u>Reason</u> $\{X \to S \text{ affine}\} \simeq \{\text{qcoh } \mathcal{O}_S\text{-algebras}\}$

+ above descent for qcoh modules compatible w/ $\otimes$

<u>Variants</u> Descent for affine group schemes, torsors under affine group schemes, ...

⚡ It is NOT true that schemes satisfy fpqc (or even étale) descent, cf. [BLR §6.7] or [Stacks 08KE] for examples.

(This is the motivation for the defn. of algebraic spaces.)

(Schemes satisfy Zariski descent, of course.)

§ Descent for schemes

Def $S' \xrightarrow{p} S$ fpqc. Category of descent data

1) $Des_{S'/S} := \{ (X', \phi),\ X'/S' \text{ scheme},\ \phi : p_1^* X' \xrightarrow{\cong} p_2^* X'$
s.th. cocycle holds $\}$

2) Descent datum (X', ϕ) effective $\overset{=}{\text{def}}$ lies in essential image.

Example Assume $\exists$ section $S \xrightarrow{s} S'$. Then every descent datum is effective: $(X', \phi) = p^* s^* X'$

Recall X quasi-affine $\overset{=}{\text{def}}$ X qc + isomorphic to open in affine

$\iff X$ qc + $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ open immersion.

($X \xrightarrow{f} S$ quasi-affine $\overset{=}{\text{def}}$ f qc + $\forall$ affine open $U \subseteq S$ affine open, $f^{-1}(U)$ quasi-aff.)

Thm [BLR Thm 6.1.6]

1) $p^* : Sch/S \longrightarrow Des_{S'/S}$ fully faithful.

2) quasi-affine descent data are effective.

3) Assume S', S affine. Then (X', ϕ) effective $\Leftrightarrow$

$\Leftrightarrow$ $\exists$ open covering of X' by ϕ-stable quasi-affines. (Equivalent: ϕ-stable affines.)

Here $U' \subseteq S'$ ϕ-stable $\Leftrightarrow$ ϕ restricts to $p_1^* U' \xrightarrow{\cong} p_2^* U'$

(Rmk Logic here same as for construction of quotients for finite group actions: First construct $\operatorname{Spec} A/G$. Then extend to all $G \curvearrowright X$ s.th. X covered by G-stable affine opens.)

<u>Sketch</u> 1) & 3) omitted.

[Stacks 0246]

2) By fully faithfulness from 1), may work locally on S, i.e. assume S affine.

Replace S' by $\coprod_{i=1}^{r} U_i'$ for $U_i' \subset S'$ affine, $\bigcup U_i' \to S$ surjective (fpqc property) $\Rightarrow$ wlog S' affine.

Then $X' \hookrightarrow T' := \operatorname{Spec} A'$, $A' = \Gamma(X', \mathcal{O}_{X'})$ quasi-affine.

Since $X' \to S'$ is qc, $\Gamma(p_i^* X', \mathcal{O}) = p_i^* A'$.

ϕ provides descent datum $p_1^* A' \xrightarrow{\cong} p_2^* A'$.

Effective, yields affine $T = \operatorname{Spec} A/S$.

$\Rightarrow \quad X' \subseteq \left(p^* T = S' \times_S T \, , \, \phi \right)$

qc ϕ-stable open subset of descent datum.

<u>Claim 1</u> $|X'| = p^{-1}(|X|)$ for subset $|X| \subseteq T$.

<u>Proof</u> Assume (s_1', t), $(s_2', t) \in S' \times_S T$ (scheme valued points)

where $(s_1', t) \in X'$

In particular, s_1' & s_2' lie above same point in S.

Then $(s_1', s_2', t) \in p_1^* X'$.

ϕ-stability $\Rightarrow \phi(s_1', s_2', t) = (s_1', s_2', t) \in p_2^* X'$

$\Rightarrow (s_2', t) \in X'$. $\square$

<u>Left to show</u>

<u>Claim 2</u> $|X|$ is open.

<u>Proof</u> $S' \to S$ flat, qc, surjective

$\Rightarrow \quad S' \times_S T \to T$ flat, qc, surjective.

<u>Prop</u> $f : T' \to T$ flat qc surjective

$\Rightarrow f$ <u>submersive</u> ie.

$U \subset T$ is open $\iff f^{-1}U$ is open.

(Means that T has quotient topology.)

<u>Proof</u> Local on T, so wlog $T = \operatorname{Spec} A$.

$T' = \bigcup_{i=1}^{r} T_i$ affine open covering.

Then $f^{-1}U$ open $\iff f^{-1}U \cap T_i$ open $\forall i$

$\Rightarrow$ wlog $T' = \bigsqcup T_i = \operatorname{Spec} B$

Assume $f^{-1}(Z) = \operatorname{Spec} B/J$ closed.

Surjectivity of $f \Rightarrow Z = f(f^{-1}(Z))$

$= \text{Image of } \operatorname{Spec}(A \longrightarrow B/J)$

<u>Fact</u> Flat maps are generalizing (= going down)

($R \to S$ flat, $\mathfrak{p} \subseteq \mathfrak{p}' \subseteq R$, $\mathfrak{q}' \subseteq S$ above $\mathfrak{p}'$

$\Rightarrow R_{\mathfrak{p}'} \longrightarrow S_{\mathfrak{q}'}$ faithfully flat, in partic surjective.)

$\Rightarrow \operatorname{Spec} A \setminus Z = f(\operatorname{Spec} B \setminus f^{-1}(Z))$

stable under generalizations

$\Rightarrow Z$ stable under specializations.

<u>Final argument</u> This implies Z closed.

$I = \ker(A \to B/J)$. Then Z contains all minimal primes of $\operatorname{Spec} A/I$.

Indeed, $\mathfrak{p}$ such a prime. Then $\operatorname{Spec}(A/I)_{\mathfrak{p}} = \{\mathfrak{p}\}$, and $(A/I)_{\mathfrak{p}} \to (B/J)_{\mathfrak{p}}$ injective by exactness of localization. So $(B/J)_{\mathfrak{p}} \neq 0$.

$+$ Z stable under specialization $\Rightarrow$ $Z = V(I)$. $\square^3$

<u>Example</u> $X/\operatorname{Spec} k$ curve. Then $\coprod_{x \in X} \operatorname{Spec} \mathcal{O}_{X,x} \to X$

flat + surjective. Not qc and, indeed, X does not carry quotient topology.

Question: How to conveniently ensure existence of ϕ-stable (quasi-) affine coverings?

Answer: Enrich descent datum by ample line bundle.

§ Descent w/ ample line bundles [Stacks II:28.26 II:29.37]

<u>Def</u> $\mathcal{L}$ on X <u>ample</u> $\overset{=}{\text{def}}$ X quasi-compact &

1)

$\forall x \in X \ \exists n \geq 1, \ s \in \Gamma(X, \mathcal{L}^{\otimes n})$ s.th.

$x \in X_s := D(s)$ & X_s affine.

2) $f : X \to S$, $\mathcal{L}$ on X <u>f-ample</u>

$\overset{=}{\text{def}}$ $\exists$ open affine covering $S = \cup S_i$ s.th. $\mathcal{L}|_{f^{-1}(S_i)}$ ample.

<u>Rmk</u> Existence of an ample l.b. $\Rightarrow$ X resp. f separated.

<u>Construction</u> X qc, $\mathcal{L}$ lb on X.

$A := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$ graded ring

$U := \cup X_s$, $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ $n \geq 1$.
(ie $s \in A_+$)

<u>Canonical</u> $\psi : U \to \operatorname{Proj}(A)$

$x \mapsto$ ideal of s s.th. $s(x) = 0$.

Analogous $f: X \to S$ qc, $\mathcal{L}$ on X

$\mathcal{A} := \bigoplus_{n \geq 0} f_*(\mathcal{L}^{\otimes n})$

$U = \bigcup_{n \geq 1} \{ s \mid f^* f_* (\mathcal{L}^{\otimes n}) \longrightarrow \mathcal{L}^{\otimes n} \text{ surjective in } s \}$

$\Rightarrow \exists\ \psi : U \longrightarrow \underline{\mathrm{Proj}}_{\mathcal{O}_S} \mathcal{A}$.

<u>Prop</u> [Stacks 01VJ]

1) $\mathcal{L}$ ample (resp. f-ample)

2) X quasi-separated (resp. f q.s.),

$U = X$ & ψ open immersion.

Let now again $S' \to S$ fpqc, (X', ϕ) descent datum.

Compatible descent datum for line bundle $\mathcal{L}'$ on X'

$\underset{\text{def}}{=}$ $\lambda : p^* \mathcal{L}' \xrightarrow{\cong} \phi^* p^* \mathcal{L}'$ (on $p_1^* X'$)

s.th. cocycle cond holds:
(on $S'' \times_{p_1, S'} X'$)

$$\begin{tikzcd}
p^* \mathcal{L}' \arrow[r, "p_{12}^* \lambda"] \arrow[dr, "p_{13}^* \lambda"'] & (p_{12}^* \phi)^* p^* \mathcal{L} \arrow[d, "(p_{12}^* \phi)^* (p_{23}^* \lambda)"] \\
& (p_{13}^* \phi)^* p^* \mathcal{L} = (p_{12}^* \phi)^* (p_{23}^* \phi)^* p^* \mathcal{L}
\end{tikzcd}$$

Easier $\mathcal{L} \longmapsto$ total space $L = \underline{\mathrm{Spec}}_{\mathcal{O}_S} \bigoplus_{n \geq 0} \mathcal{L}^{-n}$.

Then
$$\begin{tikzcd} \lambda : S'' \times_{p_1, S'} L \arrow[r, "\cong"] \arrow[d] & S'' \times_{p_2, S'} L \arrow[d] \\ \phi : S'' \times_{p_1, S'} X \arrow[r, "\cong"] & S'' \times_{p_2, S'} X \end{tikzcd}$$

& cocycle condition for λ is same as for schemes.

Thm (Grothendieck) $S' \to S$ fppqc, $X' \xrightarrow{f'} S'$ qc + $\mathcal{L}'$ f'-ample. $(X', \mathcal{L}', \phi)$ descent datum.

Then $(X', \mathcal{L}', \phi)$ effective, ie there is (unique up to iso) $X \xrightarrow{f} S$ + $\mathcal{L}$ f-ample s.t.

$$(X', \mathcal{L}', \phi) \cong p^*(X, \mathcal{L}).$$

Sketch S, S' affine.

λ induces descent datum for $\mathcal{A}' = \bigoplus_{n \geq 0} f'_*(\mathcal{L}')^{\otimes n}$.

Descends to qcoh graded $\mathcal{O}_S$-algebra $\mathcal{A}$.

Given $x' \in X'$, $\exists n$ & $s' \in \Gamma(X', \mathcal{L}^{\otimes n})$

s.th. $x' \in X'_{s'}$. (use $\mathcal{L}$ ample)

Can write $s' = \sum a_i \otimes s_i$ with $s_i \in \mathcal{A}_n$

Then $s'(x') \neq 0 \Rightarrow \exists i$ with $s_i(x) \neq 0$.

Since $\lambda(1 \otimes s_i) = 1 \otimes s_i$, X'_{s_i} is ϕ-stable.

Moreover, X'_{s_i} quasi-affine since it equals

$$\psi(X') \cap \underbrace{D_+(s_i)}_{\text{affine} \subseteq \operatorname{Proj} \mathcal{A}'} \quad + \ \psi \text{ is qc open immersion.}$$

$\square$

§ Application to ECs

Prop $S' \to S$ fpqc. Any descent datum of elliptic curves $(E', \phi)/S'$ is effective.

Proof $\mathcal{L}' := \mathcal{O}_{E'}([e'])$ relatively ample.

$\phi : p_1^* E' \to p_2^* E'$ map of ECs

$\Rightarrow \quad \phi^*(p_2^*[e']) = p_1^*[e']$ in sense of equality of closed subschemes.

In other words, the natural map

$$\begin{array}{ccc} \phi^* \mathcal{O}_{p_2^* E'} & \xrightarrow{\phi^*} & \mathcal{O}_{p_1^* E'} \\ \cup & & \cup \\ \phi^* \mathcal{I}_{p_2^*[e']} & \xrightarrow{\cong} & \mathcal{I}_{p_1^*[e']} \end{array}$$

Its inverse provides a descent datum for $\mathcal{L}'$.

(Cocycle condition follows from that of ϕ.) $\square$

Other examples

1) Families C/S of curves of genus $g \geq 2$

($\Omega^1_{C/S}$ is canonical ample line bundle)

Same with $g = 0$, take $(\Omega^1_{C/S})^{-1}$.

2) Pairs (A, λ), A/S projective abelian scheme

+ $\lambda : A \longrightarrow A^\vee$ polarization

satisfy descent. (λ roughly corresponds to ample line bundle.)

3) Abelian schemes themselves at least satisfy étale descent.

(Group algebraic spaces are schemes.)